*Proc. Natl. Acad. Sci. USA*
Vol. 96, pp. 10568–10571, September 1999

# Perspective

# Chaos, but in voting and apportionments?

*Donald G. Saari**

*Department of Mathematics, Northwestern University, Evanston, IL 60208-2730*

**Mathematical chaos and related concepts are used to explain and resolve issues ranging from voting paradoxes to the apportioning of congressional seats.**

Although the phrase "chaos in voting" may suggest the complexity of political interactions, here I indicate how "mathematical chaos" helps resolve perplexing theoretical issues identified as early as 1770, when J. C. Borda (1) worried whether the way the French Academy of Sciences elected members caused inappropriate outcomes. The procedure used by the academy was the widely used plurality system, where each voter votes for one candidate. To illustrate the kind of difficulties that can arise in this system, suppose 30 voters rank the alternatives *A, B, C*, and *D* as follows (where "$>$" means "strictly preferred"):

Table 1. Thirty-voter example

| Number | Ranking |
|---|---|
| 3 | $A > C > D > B$ |
| 6 | $A > D > C > B$ |
| 3 | $B > C > D > A$ |
| 5 | $B > D > C > A$ |
| 2 | $C > B > D > A$ |
| 5 | $C > D > B > A$ |
| 2 | $D > B > C > A$ |
| 4 | $D > C > B > A$ |

The $A > B > C > D$ plurality outcome, where only top-ranked candidates are voted for, suggests that *A* is the voters' candidate of choice. But is she? If any candidate or pair drops out, the new plurality ranking flips to agree with the reversed $D > C > B > A$. For example, if *C* drops out, the $D > B > A$ outcome has a 11:10:9 tally. If *B* and *D* drop out, the $C > A$ outcome has a 21:9 tally. It is arguable that *D*, not *A*, is the candidate of choice, even though *D* is plurality bottom ranked.

This disturbing election change as candidates leave is but one difficulty. For another, instead of considering just each voter's top-ranked candidate, include more ranking information. A way to do so is with a positional method, where points are assigned to alternatives according to how voters rank them. To illustrate with four candidates, the assigned weights are given by a voting vector $\mathbf{w}^4 = (w_1, w_2, w_3, 0)$, $w \geq w_{j+1}$, $w_1 > 0$. Surprisingly, the outcome can depend on the voting method. Indeed, with Table 1 preferences, each alternative wins with an appropriate choice of $\mathbf{w}^4$. To illustrate with a simpler 10-voter profile (a profile lists all voters' preferences):

Table 2. Ten-voter example

| Number | Preference |
|---|---|
| 2 | $A > B > C > D$ |
| 1 | $A > C > D > B$ |
| 2 | $A > D > C > B$ |
| 2 | $C > B > D > A$ |
| 3 | $D > B > C > A$ |

*A* wins with the plurality vote (1, 0, 0, 0), *B* wins by voting for two candidates, that is, with (1, 1, 0, 0), *C* wins by voting for three candidates, and *D* wins with the method proposed by Borda, now called the Borda Count (BC), where the weights are (3, 2, 1, 0). Namely, election outcomes can more accurately reflect the choice of a procedure rather than the voters' preferences. This aberration raises the realistic worry that inadvertently we may not select whom we really want.

"Bad decisions" extend into, say, engineering, where one way to decide among design (material, etc.) alternatives is to assign points to alternatives based on how they rank over several criteria. A "best of the best" approach, selecting the alternative that is top ranked over more criteria, is equivalent to the plurality vote (1, 0, 0, 0). A conservative approach of selecting the alternative that is bottom ranked over the fewest criteria is equivalent to (1, 1, 1, 0). By interpreting "voters" as "criteria," Table 2 proves that engineering decisions can reflect the procedure rather than carefully assembled data.

## The U.S. Supreme Court

A second "chaotic" concern is where apportionments are made proportionally according to collected data; this includes the proportional voting methods widely used in Europe and South America, where the number of seats a political party wins in an election is proportional to the number of votes they receive. For simplicity, emphasis is placed on the closely related problem, where the apportionment of congressional seats to states in the U.S. is done according to census figures. The political importance of this assignment of representatives has generated controversy leading to a recent U.S. Supreme Court decision about statistical census techniques. But other mathematical difficulties most surely will generate Supreme Court cases during the next decade. The problem arises because the states' exact apportionments usually involve fractions; how should they be rounded off? One approach assigns each state the integer value of its exact apportionment and assigns remaining seats according to the fractional parts. In the Table 3 three-state example and house size of 10, the integer values assign nine of the ten seats. State A receives the extra seat because it has the largest decimal remainder.

With house size 11, however, the *A, B, C* exact apportionments are, respectively, 2.607, 3.663, and 4.73, so *A* loses representation, while *B* and *C* gain with the 2, 4, 5 apportionment. Rather than an amusing exercise, this example characterizes problems where several states, such as Alabama and Maine, lost seats with increases in the number of representatives; it is related to George Washington's first Presidential veto over competing methods of handling these apportionment difficulties; and it is why Congress has 435 seats. (The house size was adjusted to 435 to avoid just this mathematical peculiarity with the 1910 census.)

*To whom reprint requests should be addressed. E-mail: dsaari@nwu.edu.

Table 3. Three-state apportionment

| State | Population | Exact apportionment | Integer | Apportionment |
|---|---|---|---|---|
| *A* | 237 | 2.37 | 2 | 3 |
| *B* | 333 | 3.33 | 3 | 3 |
| *C* | 430 | 4.30 | 4 | 4 |
| Total | 1,000 | 10 | 9 | 10 |

## Voting Procedures

For over two centuries, problems caused by dropping candidates from the voting list or changing voting procedures have created an enormous rapidly growing literature (www.maxwell.syr.edu/maxpages/faculty/jskelly/biblioho.htm). For example, it was discovered that the plurality outcome can be reversed by dropping the bottom-ranked candidate, or that two different procedures can have reversed outcomes with the same voter preferences (2). The introductory examples prove that much more can go wrong.

Although important, positional procedures (that is, ballots that are tallied with a specified voting vector) have proved to be formidable to analyze. The underlying complexity is suggested by how societal outcomes can change by dropping alternatives, or by how these procedures can generate millions of different election rankings with one 10-candidate profile (3): once the ballots are marked, the voters' opinions remain fixed, but varying the tallying method generates millions of contradictory outcomes, where each alternative wins with some procedures but is bottom ranked with others. So, which conflicting outcome reflects the "voters' opinions?" The complexity is further underscored by K. Arrow's (4) seminal result suggesting that all nondictatorial voting procedures have flaws. A natural reaction is a (5) resigned attitude that "[t]he choice of a positional voting method is subjective."

To respond, a first goal is to identify all possible paradoxes that can occur with any profile. Do the above examples essentially catalogue all oddities that can occur with vote tallying procedures, or can more convoluted changes in election outcomes occur? Beyond the complexity of the standard combinatoric analysis, what hinders achieving this goal is the nature of a "paradox"—a counterintuitive outcome. Namely, if the goal is to discover everything we do not expect to occur, what do we look for? "Chaos" helps overcome these fundamental difficulties.

## Chaos

In using chaos, dynamical aspects such as "homoclinic points" and "fractals" do not appear. Instead, concepts of chaos and symbolic dynamics are modified to create new techniques. To indicate the approach, the iterative dynamical system $\mathbf{p}_{n+1} = f(\mathbf{p}_n)$, with starting point $\mathbf{p}_0$, defines trajectory $(\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n, \ldots)$. By partitioning phase space into regions, for example *A, B*, . . . , *Z*, an associated sequence—a word—is created by replacing each $\mathbf{p}_j$ with a symbol representing its containing region. For instance, the word (*A, C, D, B, B*, . . . ) starts with $\mathbf{p}_0$ in region *A*, $\mathbf{p}_1$ in *C*, . . . . As different words capture different dynamical behaviors, a goal is to identify all words. Each word proves that a starting point $\mathbf{p}_0$ exists whose orbit has the specified qualitative behavior (see textbooks, refs. 6, 7).

This objective of finding all words is similar to the goal of finding all voting paradoxes. Replacing $\mathbf{p}_j$ is the election tally of the *j*th subset of candidates. Instead of partitioning phase space, the symbols represent rankings of each subset of candidates. What replaces $\mathbf{p}_0$ is a profile, for example, such as Tables 1 and 2. A "word" becomes a listing of election rankings, one for each subset of candidates; it establishes the existence of a profile where each subset's election outcome is as specified. By using techniques motivated by concepts from chaos, all paradoxes now can be identified. The conclusion as captured by the following theorem is most discouraging for a democracy.

THEOREM 1 (8): *For $n \geq 3$ alternatives, select a ranking for each of the $2^n - (n+1)$ subsets of two or more alternatives. There exists a profile where each subset's sincere plurality election ranking is the selected ranking.*

According to this assertion, anything can happen in the same voters' plurality election rankings when different subsets of candidates are considered. There is a profile, for instance, where the plurality ranking of each subset with an even number of alternates agrees with $A > B > \ldots > Z$, but when the same voters sincerely rank a subset with an odd number of alternatives, the outcome flips to agree with $Z > \ldots > A$. Even worse, as a way to prove that there need not be any relationship among election rankings, *Theorem 1* allows us to use a random number generator to select the ranking for each subset of candidates. Then *Theorem 1* ensures that a profile exists where each subset's sincere election ranking is the randomly generated one. Thus, *Theorem 1* is a worrisome chaotic conclusion about our widely used election tool of the plurality vote—and about decisions we have made.

With all its lack of consistency in election outcomes, it is reasonable to wonder whether the plurality vote can be improved on. A way to examine this question is to characterize all outcomes for all ways to tally the ballots for each subset of candidates. The news is as discouraging as the results from *Theorem 1*; in general, anything can happen.

THEOREM 2 (8, 9): *For $n \geq 3$ alternatives, list the subsets of two or more alternatives as $S_1, \ldots, S_{2^n-(n+1)}$. Assign to each $S_j$ a voting vector to tally the ballots and a ranking. For almost all choices of voting vectors (that is, with the exception of an algebraic set in an appropriate space), there exists a profile so that each subset's sincere election ranking with the specified procedure is the selected ranking.*

Although *Theorem 2* requires almost all procedures to suffer the troubles of the plurality vote, hope comes from the assertion that a lower dimensional set of methods avoid some paradoxes. So the next step is to identify the procedure that minimizes the number and kinds of paradoxes. The answer is the BC where, for *n* alternatives, $n-1, n-2, \ldots, 0$ points are assigned to a voter's first-, second-, . . . , last-ranked alternative.

THEOREM 3 (8, 10): *For $n > 3$ alternatives, use the BC to tally each subset. Each BC word (that is, each listing of BC election rankings over the subsets of alternatives) also occurs with all other ways to tally election outcomes. All other voting vectors admit words (election paradoxes) that cannot occur with the BC.*

Only the BC minimizes problems and maximizes consistencies in election outcomes over the subsets of candidates. Indeed, if a profile creates troubling BC election rankings over the subsets, *Theorem 3* ensures for all other ways to tally ballots, some profile exhibits the same troubling outcomes. All other methods, however, admit election inconsistencies never suffered by the BC. Thus, only the BC maximizes the numbers and kinds of positive relationships. As a brief sample (10–12), only BC's election rankings must be related to the pairwise outcomes; for any other positional procedure, the election outcome can even reverse the pairwise outcomes. Only BC's election ranking for all candidates must be related to its election ranking for all subsets of *k* candidates; $k = 2, \ldots, n-1$.

The number of bothersome election paradoxes identified by these theorems quickly reaches the billions with only five and six alternatives. All evidence indicates that these serious concerns cannot be dismissed as coming from rare concocted examples. Instead, the theorems describe robust events that remain even after supporting profiles are perturbed (that is, in an appropriate space, the profiles defining strict outcomes are open sets). With three candidates and conservative assumptions about voters' preferences, about 69% of the preferences allow outcomes to change with the voting procedure (13); the evidence shows that the likelihood of problems rapidly increases with the number of alternatives. Indeed, by knowing what causes the paradoxes, one can identify many actual elections where it is arguable that the "winner" does not reflect the voters' beliefs.

### Explanations and Strategic Voting

Now that the extent of election paradoxes is identified by the above theorems, we need to explain each of them, to characterize all supporting profiles, and to identify all relationships. This project has recently been completed. (For $n = 3$, see ref. 14; for $n \geq 3$, see ref. 12.) These results, which depend on how structures of "chaos" are modified to analyze voting, statistics, probability, and so on, involve symmetry groups. A natural symmetry is "neutrality;" for example, if all voters confused A and B, then the outcome is corrected by attaching the proper name to each tally. Namely, election procedures respect permutations of the names of the alternatives. Although this makes the permutation group a natural symmetry, each subset has its own permutation group. A "wreath product of permutation groups" captures the relationship among them.

This structure suggests that refined properties arise by determining how various subsymmetries affect different procedures. For instance, with $n$ candidates, the Condorcet profile (15) for $n$ voters is as follows. Start with a first voter's ranking, say $A > B > \ldots > Z$. Move each candidate up one position and place the former top-ranked candidate at the bottom; this defines $B > \ldots > Z > A$. Continue until rankings for $n$ voters are defined. As this $Z_n$ symmetry orbit treats each candidate equally (each is in each position exactly once), no candidate is favored over another; indeed, all positional outcomes are complete ties. The pairwise vote, however, has a $A > B$, $B > C, \ldots, Z > A$ cycle with impressive $(n - 1)$:1 tallies. The source of this cycle, which has generated a large literature, now is apparent; pairwise comparisons lose crucial information about the profile's full symmetry. Indeed, it is the Condorcet portions of a profile that force the pairwise outcomes to differ from the positional ones.

Similarly, by identifying all appropriate subgroup behavior, all possible paradoxes now can be identified and explained, and all supporting profiles characterized. The conclusion is that when a procedure cannot recognize the symmetry of certain portions of a profile, paradoxes emerge. Only the BC is spared these difficulties; it avoids them because of the fixed difference between BC successive weights.

These results also prove that the strange election behavior documented by the theorems are the rule rather than the exception. This is in keeping with earlier research, where preferences are selected in multidimensional spaces (representing several issues), which proves that for almost all profiles, our commonly used majority vote suffers severe problems (16–19). This conclusion holds even if a victory requires more than half of the voters (20).

Finally, recall the sensitivity of chaos to initial conditions, where small changes in starting positions can force significantly different behavior; that is, a different word. In voting, the many behavioral interpretations for a slightly changed profile (which defines a new outcome) include strategic voting (21, 22), or where a voter obtains a personally better outcome by abstaining (for example, ref. 23), and so forth. These results normally require specialized arguments, but by mimicking the concepts behind the sensitivity results in chaos, a unified argument and simple tool (11) emerges that quickly analyzes these properties for any procedure and allows us to identify where, when, and why they occur.

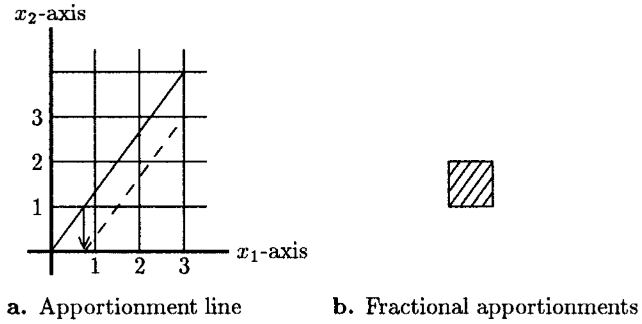


FIG. 1. Converting exact apportionments to a flow on a torus.

For instance, all reasonable procedures can be manipulated once there are three or more alternatives (21, 22). Some systems tend to give distorted outcomes (for example, the plurality vote), so voters are motivated to vote strategically (the "don't waste your vote" cry), while for others it is easy to determine how to cast a strategic vote (for example, the BC). Thus, a reasonable goal is to find the system least likely to allow a small number of the voters to successfully manipulate the outcome. The surprise, which contradicts what has been common belief for two centuries, is that for three alternatives, it is the BC (11, 24). This conclusion depends on the symmetry properties of the BC. In other words, again, concepts from chaos and geometry provide a much clearer picture of strengths and weaknesses of voting procedures.

### Apportionment Problems

Chaos plays an equally strong role in identifying apportionment difficulties whether they involve congressional seats, draft quotas, the distribution of benefits, and so on. Although these concerns have been studied for centuries, Huntington's (25, 26) work of the 1920s remains among the most insightful. Interest was resurrected with the Balinski and Young (27) paper, which cites some of the history and analyzes certain paradoxes (28). [For added history and discussion, see the 1992 U.S. Supreme Court apportionment decisions (29, 30)]. A geometric approach that answers many of these concerns is in ref. 11.

Analyzing apportionment methods has proved to be surprisingly difficult; this is because "rounding off" in higher dimensions approximates chaos (11). To explain, if the $j$th component of $\mathbf{p} = (p_1, \ldots, p_s)$ represents the fraction of the population of state $j$ relative to the total population, the exact apportionment is given by

$$\mathbf{x}'(t) = \mathbf{p}, \text{ or } \mathbf{x}(t) = \mathbf{p}t, \qquad [1]$$

where $t$ specifies the number of available seats—the house size. "Rounding off" uses the fractional part of each $\mathbf{x}(t)$ component. To introduce geometry, let $s = 2$, and represent Eq. **1** as the solid line in Fig. 1*a*.

To represent the fractional portion of an apportionment, the integer part needs to be dropped. To do this, notice that when the apportionment line meets a horizontal line, State 2's exact apportionment ($x_2$) is an integer; that is, the $x_2$ fractional part reverts to zero. So, to drop the integer portion, cut the

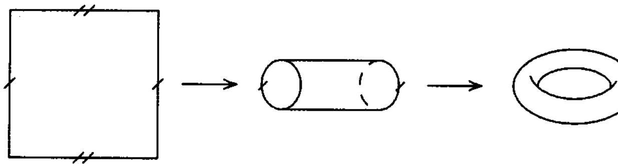

FIG. 2. Creating a torus from a square.

apportionment line and slide it to the $x_2 = 0$ axis to reflect this change (Fig. 1*a*, dashed line). Similarly, when the apportionment line meets a vertical line, $x_1$ has an integer value. To keep only the fractional portions, cut the apportionment line and translate it to the $x_1 = 0$ axis. In this manner, the disjoint lines in Fig. 1*b* represent all simultaneous fractional apportionments. The disconnected lines are reconnected in the following manner. Because the two horizontal edges of the Fig. 1*b* square represent the $x_2$ fractional value of zero, they can be glued together to create a cylinder (Fig. 2). Notice how this construction connects the lines at the top and bottom edge. Doing the same for the vertical edges shows that the fractional parts define a smooth line on a torus. Different initial conditions for Eq. **1** determine different torus lines. For the 50 U.S. states, the apportionment problem is represented by a flow on a 50-dimensional torus.

Recall from chaotic dynamics that a simple example of chaos is this flow on a torus where $p_2/p_1$, from $\mathbf{p} = (p_1, p_2)$, is an irrational number (for example, ref. 6). The motion is not "chaotic" when each $p_j$ is a fraction, as with apportionments, but enough chaotic features remain to cause most apportionment difficulties. For intuition, consider a closely related system of two children on swings. With the same frequency, they define a fixed pattern. But, the greater the incommensurability of the frequencies, the greater the eventual differences in their positions. Indeed, arbitrarily select a position for each swing; eventually, each swing will simultaneously be arbitrarily close to the indicated location. The same is true for fractional portions of the exact apportionments; with sufficient disagreement among the ratios of the $p_j$s, the fractional portions eventually come arbitrarily close to any designated point on the torus—either the 2-dimensional torus of the example, or the 50-dimensional one representing the U.S.

To understand the significance, notice how the Table 3 behavior requires a small state's fractional part to justify an extra seat, but the next house size increases the fractional terms of larger states. As an open set of fractional values exhibits this paradox, the "near-chaotic behavior" of apportionments, or other rounding-off problems, ensures this behavior most surely occurs.

THEOREM 4 (11): *For $n \geqq 3$ states and almost all populations figures and any choice of initial conditions, there exists a house size where at least one state loses a representative as the house size increases.*

Even more worrisome apportionment problems can occur. Based on a National Academy of Sciences (31) study, the U.S. has adopted a method to avoid these difficulties; closely related procedures are used in other countries to determine the division of seats to political parties when proportional voting approaches are used. Start with house size 50 (one representative per state) and successively add a representative up to 435 by assigning each seat to a state based on a "fairness" criterion. The peculiarities of higher dimensional spaces (11) ensure other troubling paradoxes. For instance, if a state's exact apportionment is 23.2, we expect 23 or 24 seats; but our procedure can allow an allocation of, say, 18 or 27. Again, with "chaotic dynamics" arguments (using different units), it can be shown that these are serious concerns. Fortunately, these problems have not occurred with current population figures. Nevertheless, "chaotic" and geometric arguments prove that with a high likelihood and with each census, our procedure creates perceived inequities—settings that most surely will generate new U.S. Supreme Court cases.

This research was supported by National Science Foundation Grant DMI-9971794 and an Arthur and Gladys Pancoe Professorship.


1. Borda, J. C. (1781) *Mémoire sur les élections au scrutin*. *Histoire de l'Académie Royale des Sciences* (Académie Royale des Sciences, Paris).
2. Fishburn, P. (1981) *Discrete Appl. Math.* **3,** 27–36.
3. Saari, D. G. (1992) *Soc. Choice Welfare* **9,** 277–306.
4. Arrow, K. J. (1963) *Social Choice and Individual Values* (Wiley, New York), 2nd Ed.
5. Riker, W. (1982) *Liberalism Against Populism* (Freeman, San Francisco).
6. Devaney, R. (1989) *An Introduction to Chaotic Dynamical Systems* (Addison–Wesley, Reading, MA).
7. Robinson, C. (1995) *Dynamical Systems* (CRC Press, Ann Arbor).
8. Saari, D. G. (1989) *J. Econ. Theory* **48,** 443–475.
9. Saari, D. G. (1995) *SIAM Rev.* **37,** 37–52.
10. Saari, D. G. (1990) *Soc. Choice Welfare* **7,** 279–317.
11. Saari, D. G. (1995) *Basic Geometry of Voting* (Springer, Heidelberg).
12. Saari, D. G. (2000) *Econ. Theory* **15,** 1–103.
13. Saari, D. G. & Tataru, M. (1999) *Econ. Theory* **13**, 345–363.
14. Saari, D. G. (2000) *J. Econ. Theory*, in press.
15. Condorcet, M. (1785) *Essai sur l'Application de l'Analyse à la Probabilité des Décisions Rendues à la Pluralité des Voix* (Imprimerie Royale, Paris).
16. McKelvey, R. D. (1976) *J. Econ. Theory* **12,** 472–482.
17. McKelvey, R. D. (1979) *Econometrica* **47,** 1086–1112.
18. Cohen, L. (1979) *J. Econ. Theory* **20,** 1–12.
19. Schofield, N. J. (1983) *Rev. Econ. Studies* **50,** 695–705.
20. Saari, D. G. (1997) *Econ. Theory* **9,** 219–262.
21. Gibbard, A. (1973) *Econometrica* **41,** 587–601.
22. Satterwaite, M. (1975) *J. Econ. Theory* **10,** 187–217.
23. Moulin, H. (1988) *J. Econ. Theory* **45,** 53–63.
24. Saari, D. G. (1990) *Public Choice* **64,** 21–41.
25. Huntington, E. V. (1921) *Proc. Natl. Acad. Sci. USA* **7,** 123–127.
26. Huntington, E. V. (1928) *Trans. Am. Math. Soc.* **30,** 85–110.
27. Balinski, M. & Young, H. P. (1975) *Am. Math. Monthly* **82,** 701–730.
28. Balinski, M. & Young, H. P. (1982) *Fair Representation* (Yale Univ. Press, New Haven, CT).
29. *Sec. of Commerce v. Massachusetts,* US Reports; 91-1502 (1992).
30. *Montana v. US Dept. of Commerce*, US Reports; 91-860 (1992).
31. Bliss, G., Brown, E. W., Eisenhart, L. P. & Pearl, R., *Report to the President of the National Academy of Sciences*, February 9, Congressional Record, 70th Congress, 2nd Session, 70 (1929) 1015–1047.